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THE TIME CONCEPT I.

BY

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1. The analysis of the concept of time presents peculiar difficulties. To get away from the subjective aspect is the scientist's main purpose; but when he attempts to approach it from the objective standpoint he finds that it is so inseparably connected with the Concept of Space that the process of abstraction becomes one of almost insuperable difficulty. The object of this article is to make an attempt to clarify the Concept, and although the writer knows that he will not be wholly successful, he still believes that some useful purpose will have been achieved.

2. According to the celebrated definition of Aristotle (*Physic* IV c. 11.) time is "The number (numeration) of motion according to priority and subsequence". By "number" he meant "measure", for the latter may always be expressed as a number. "Motion" for the Philosopher signifies not only local motion, but any sort of a change from one state to another. By the concluding phrase it is implied that motion, movement, or change consists of parts successively flowing, of which one is always prior and another posterior.

2.1 Time is not any particular motion, nor is it motion in the abstract; it is, however, conceived by Aristotle as itself having motion or movement. Time accompanies movement not so much in virtue of the movement as in virtue of the succession that we find in movement and the priority and subsequence that we detect in that succession. Moreover, time is measurable, and as such can be put into correspondence with the system of arithmetical numbers. This is a one-sided Concept of time, which will serve as the starting point of our analysis.

3. *Change.* A "change", "motion" or "movement", we shall take to mean a difference in the state or condition perceived by some observer in some object of perception: whether it be the heat of the day, or his own appetite, or water boiling, or the position of

the hands of a clock. A motion, however, is not the alteration in the consciousness of the observer, but the alteration in the state of external things, to which the observer, and anybody else would attribute the change in the observer himself. The change we are considering is so far independent of the observer, that it corresponds to an external reality although observed by the observer. (The present article is written for "moderate realists", and the writer does not wish to digress into the relative merits of "realism" and "idealism".)

3.1 In the observation of any change there is a succession of perceptive acts; and the order of the succession of observables is independent of the observer. Whether this be the experience of the human race or not, we cannot get on without it as an *axiom* for scientific purposes, apart from philosophical disputations; and as a matter of fact, not only for science but for everyday practical purposes, the assumption—if assumption it be—is made.

3.2 We take it then as an *axiom* that there is *a natural order of priority and subsequence* in our acts of perception; which order is independent of the act of perceiving or of the observer, and therefore inherent in the succession of external stimuli affecting our senses. The terms themselves "prior" and "subsequent" cannot be defined, and any attempt at definition would involve a vicious circle. We may therefore take it for granted that priority and subsequence in the experience of any one observer are fixed by Nature, independently of the acts of observation. (The mathematical definition of Robb will be noted in Part II.)

4. A change may be classed as a "*flux*" or as a "*discontinuous change.*" By a "flux" we mean a continuous change in the sense that there is a succession of parts of the change such that the end of one part is the beginning of the next. A flux is capable of having parts, at least two, each part also a flux; so that we may say that each part is not the whole change, and each part is not any other part, but is exclusive of it.

4.1 In all our experience with concrete things we cannot adduce for certain one example of a flux. We are forced to admit that what we for centuries considered as fluxes have been erroneously considered as such. In this respect the Atomists were right. The change in water from the liquid to the gaseous state, or in the disintegration of the molecule, are examples of change with no "parts"; there being no intermediate stages, and stages in between

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these and so on interminably. Such changes we call "*discontinuous.*" A flux then is an idealisation of a change. It is conceived as a change, *continuous* in the mathematical sense, much in the same way as the straight line according to Dedekind is continuous, or as the Continuam of Real numbers is continuous. This is an "*axiom*" of the flux, or a "definition", in the mathematical sense, whichever one might prefer to call it.

4.2 A flux can be *measured*, and the result put into correspondence with the number system, *provided* that we can give a meaning for a flux "*segment*", or "*interval*", and for the "equality of two such intervals". With this proviso then a flux is subject to "quantity" or rather "magnitude in the mathematical sense".

5. *Simultaneity of events.* Any conscious act of perception by me can be put in the natural order with all my other perceptions as regards priority and subsequence. We need merely speak now of the acts of perception themselves apart from their remote causes, the order of which we shall not now consider. Moreover, we shall not take into consideration lapses of memory and other accidents. It is sufficient that the perception *can* be put into relation with some or a few other definite acts of perception by me, *not according to my choice, the order being itself determined apart from me.* Now consider two definite and distinct acts of perception say, of flashes of light, which are such that I am unable to place them in such a way that one is definitely prior or subsequent to the other. Whether in a particular case this be a subjective disability or not does not matter. (I have heard of two sportsmen who both fired at an animal, and not only the two of them but also two bystanders disputed—with apparent sincerity—as to who fired the first shot; everybody agreeing that the shots were not fired simultaneously.) I have two mental impressions, then, which I *must place together* in that order of succession according to priority and subsequence into which my mental impressions naturally fit. From this consideration I abstract or "isolate" the notion of *simultaneity* of perceptions and of events in general; not that I should consider events outside me as simultaneous when perceived together, but that I should extend the notion of simultaneity to events apart from considerations of perception. We are concerned at present with the *notion* of simultaneity of events not with the *determination* of simultaneity. (See any text-book on "Relativity" for the latter.)

5.1 Let us take it for granted that any one observer has such a notion of simultaneity of events.

6. To any one observer there is a sequence of instants naturally ordered. These instants, considered as *elements* have the "descriptive" property of time; by which we mean just the property of being in a sequence naturally ordered according to priority and subsequence. Any two distinct elements fall in with this property; all the instants for the observer A have the common property viz. A's time. They are distinct in as much as for A they are naturally ordered.

6.1 We can next consider A's time as an elemental concept the descriptive properties being instants; *i. e.* A's time is marked by A's instants; it has A's instants as characteristic of it. A single instant is not a sufficient descriptive property. Two instants would mark A's time insufficiently. All A's instants would be needed to mark A's time sufficiently.

6.2 Thus we have three aspects of A's time: as a descriptive property, as an element, and as an aggregate of instants.

6.3 When we consider two observers A and B, it *may* happen that one of A's instants may be in common with one of B's instants,* and when this is so, the instant common to A's time and to B's time belongs to either or both. This instant has two descriptive properties. It is sufficiently described by both together, A's time and B's time, whether the latter be considered as each merely descriptive, or each as aggregates of instants, A's instants and B's instants respectively. A and B may have one only or two, or any number of instants in common; and these common instants belong to A's time as well as to B's time; but A's time and B's time may still not be one and the same. They would be so if A's instants and B's instants are all the same. To justify the term instant as a common noun, we should find some common property to any one instant of A and any one instant of B, or to all instants whatever, be they A's or B's or C's or.......... This common property we shall term Time (with a capital T). By Time then we mean a property of all instants by which they are naturally ordered among themselves. If all the instants of each and every observer be instants for a hypothetical abstract observer, whom we shall call Mr. X, then Time is the descriptive property of X's instants, it is the element whose descriptive properties are X's instants considered as descriptive, and it is the aggregate of all X's instants considered as elemental.

6.4 The above are our descriptive *Axioms* of Time. We believe that by distinguishing between the three aspects of time and of Time,

* A and B shake hands.

there will be less confusion in philosophical speculations. We also hold that to consider the descriptive axioms first before going on to quantitative ones would lead to a clearer understanding of the subject.

7. *The continuity of Time.* That time is continuous in the sense that as an aggregate of instants, any one instant can be considered as the end of the priors as well as the beginning of the subsequents is evident; but not in the sense that between any two instants there is at least one other instant. We should indeed consider it as "*discreet*", and time as an aggregate of instants for any one observer is enumerable.

7.1 Time as an element in itself, however, is conceived of as *continuous*, or as *a flux* and subject to "quantity", or rather "magnitude". Time is the whole continuum; and any Time-interval marked by two distinct instants, considered not only as descriptive but also as elemental, has integral parts, each having this same property *i. e.* each bounded by terminal instants and having integral parts. This is then an *Axiom* of Time as an element. Between any two instants of Time there is at least one other instant. An instant is not an integral part of Time in this sense, it can be considered as a part of Time only "*reductively*", inasmuch as instants mark the bounds of Time-intervals which are integral parts of Time. (In the same way we have considered the point only reductively as part of the straight line or plane or space). To the instant magnitude is not applicable. A Time-interval can be sub-divided into integral parts mutually exclusive each remaining an interval; and the integral parts, each defined by two terminal instants, are naturally ordered with respect to each other as regards priority and subsequence, and continuous in the sense also that the terminus of the prior is the beginning of the subsequent.

7.2 An interval has two bounds which are instants. By a mathematically indefinite subdivision we can arrive at the zero-interval as an infinitesimal which should be considered as an integral part of the interval, but itself having no parts, being bounded by two distinct instants with no distinct instant between them. The infinitesimal intervals summed up mathematically in their natural order make up the whole interval.

7.3 It is not necessary here to make special reference to the mathematical concept of "*length*" and of "*measure*," either in the older Riemannian sense or in the more comprehensive sense of Lebesgue.

8. The chief use we have of time is as a medium for the comparison of changes taking place in the external world; we would measure change against change; but this is not always directly possible, so Time is to be the medium of comparison, i.e. we must resort to *calculation.*

8.1 Assume that Mr. X can observe a *simultaneous set of conditions* of a body or of a system of bodies. This we shall term a "*stage*" in its change. The Time-flux should be such that we can measure all changes whatever; with it any change can be compared; so that itself can be measured by any change. When a stage of any change is put into correspondence with an instant the compound notion may be called a "*stage-instant*," or an "*event*" Events corresponding to the same instant are "*Simultaneous.*" Any change-interval, i.e. change or part of a change terminated at both ends by stages, is called an "*Occurrence.*" An occurrence is said to "occupy" or to "take place in" the time-interval determined by the instants corresponding to its end-stages. If the end-stages of two different occurrences be simultaneous in pairs the occurrences are said to be "contemporary" or to "take place in the same time."

Definition of "exist." The external reality corresponding to the stage A affected by the instant is said to "exist" in that instant. The external reality, in which there is an occurrence, whose end stages correspond to instants t_1 and t_2 is said to "exist" in the Time-interval determined by t_1, t_2.

8.2 Since instants form a well-ordered system, Time, considered as an aggregate of instants, is most suitable as a Gaussian Co-ordinate.

9. Before going on to the application of magnitude to Time, we shall consider the question of the "*reality*" of Time. Firstly we shall see whether we are to consider the instant as "intrinsic," or as "*extrinsic*" to an event, and the Time-interval as intrinsic or as extrinsic to an occurrence.

9.1 Considering any particular event there is no reason for saying that the instant is extrinsic to it; but in a statement such as: "Two events similar in every way except that they are not contemporary occupy equal intervals of time," we are certainly considering instants as extrinsic to events. Again the instant marks the end of the non-existence of stage A as well as the beginning of the existence of stage A: considered in this way it is "intrinsic" to A. It can mark the end of the existence of stage B as well as the

beginning of the non-existence of stage B: considered in this way it is "extrinsic" to B.

9.2 That the instant corresponds to something real, i.e. is an entity affecting a real physical or concrete stage of any physical change, either intrinsically or extrinsically, would follow from the natural order of priority and subsequence to which it is subject independently of any one observer and of all observers. If the instant be intrinsic to the stage then since the natural order of priority and subsequence in external stages is independent of the mind, so is that of the natural order of priority and subsequence in instants; on the other hand if the instant be extrinsic to the stage the independence from the mind of the natural order of priority and subsequence is in the instants themselves, and through them in the stages. Thus the concept "instant" is one of a real entity, the "*physical instant.*"

9.3 Just as the instant corresponds to a real entity, so also should Time, marked out by instants, correspond to a real entity, (inexorably real as we know too well!)

9.4 Time, as it is conceived in our minds is a mathematical flux or Continuum against which we measure physical changes. In these changes, there is something real from which we isolate the flux concept. As regards the *manner* of conceiving it, it is a "formalitas rationis," but as regards the entity from which the abstraction is made, it is real.

9.5 The instant and the Time-interval, considered as intrinsic to the event and the occurrence respectively, are not the subject-matter of mathematical operations. It is only as concepts corresponding to extrinsic entities that we can deal with them mathematically.

9.6 Every change is said to "occur in time"; loosely speaking to be "due to Time", as if Time were the efficient cause of the change, whereas it is the change that marks the "flow" of Time. Things subject to change are "subject to Time", or are "in Time"; and all things about us are subject to change; hence the universe of observation is in Time.

10. Since a change is a difference in the state or condition perceived by an observer in an object of perception, the difference being independent of the observer and of the perception, there is an implicit assumption that something *endures* i.e. remains the same in spite of the difference, or else we could not recognise the two states

as belonging to the same object. That something we may call the "Substratum" of changing objects. Considering all changes whatever, something remains the same in the ever-changing world. As regards myself, throughout the changes I have undergone from babyhood to my present state, that something is my "ego." It would be a digression to proceed further. Suffice it to say that in practice Nominalism is rejected, and philosophically it is untenable. The physical sciences endeavouring to answer the question: what is it that endures in a change? carry on the investigation not only with respect to any one change but also with respect to all changes of a particular type, and even more generally with respect to all changes whatever.

10.1 The implicit assumption is that in all corporeal things there is a substratum for all changes whatever, and that common thing is called "primary matter" or simply "matter." More generally "matter" is that which undergoes change, in the sense that the different entitles in a change corresponding to the different stages are called matter.

11. *Magnitude applied to Time.* In the above the chief difficulty was the distinction between the ideal and the real. As a moderate realist we have endeavoured not to lapse into pure idealism and at the same time to avoid the errors of absolute realism; so as to steer a middle course. In the application of magnitude to Time the difficulties that arise are due also to our failure to obtain a definition of the equality of Time-intervals independently of the observer, or of the observation of physical changes. It is not changes that we measure by Time, and that is our purpose of Time (Sec. 8, 8-1), but it is time that we measure by changes. Indeed we do not measure time at all; we measure change by change, or by some particular change taken as a standard. To assert that Time is uniform or flows uniformly on is to put the cart before the horse. This may follow from the definition of the equality of Time intervals, but a mathematical definition is only a mental convenience, and it is not meant to dictate to Nature how it must act.

11·1 If Time be taken discreetly i. e. as the aggregate of elemental instants then it can be merely counted or put into correspondence with the system of positive integers, and, by a simple convention, with the whole arithmetical system of positive and negative integers. We should thus not have a "measure" of Time; but the position of the instant in the ordered succession would serve our purpose.

11·2 More generally, however, Time is considered as itself elemental, continuous in the mathematical sense and as such a flux, directed from the prior to the posterior with respect to the descriptive instants, which themselves may be considered also as elemental. Hence *the mathematical instant does not exactly correspond to the physical instant.* (See nn. 7, 7·1, 4·1 and 9·2).

11·3 When two occurrences whose end-instants are A and B, and A′ and B′ respectively are such that A and A′ coincide and B B′ coincide, i. e. the end-stages are simultaneous in pairs, then we should say that they occur in the same interval of Time. This does not help us very much. But take another occurrence whose end-instants are P and Q, which is such that apart from the fact that P and A do not coincide, Q and B do not coincide, they are similar in all respects as far as we know and can know, then we should naturally conclude that the Time-interval PQ is equal to the Time-interval AB. However, as far as we do know, no two occurrences are exactly similar in all respects e. g. with respect to orientation in Space, location, motion etc. Nevertheless this does not prevent us from abstracting the notion of equality of Time-intervals. At least each observer can have the same definition which he can use so far as *his time* is concerned.*

11·4 Thus for any one observer there is a means of defining the equality of Time-intervals, so that his time is measurable, i. e. can be considered as a magnitude. The problem is: can there be a universal measure of Time?

12. As observed above we actually set off change against change, measuring one change by another or we take some change as a standard and measure all changes by it. Of all changes, change in local position is the most convenient to adopt as the standard. For we have compared Time with the Linear Continuum and since magnitude can be applied to the spatial change, it will correspond to the Time taken by the spatial change. In this way measure of Time can be made by linear measurements. The most obvious is the change in the position of the Sun in the sky with respect to the vertical, or that of the Sun in the sky with respect to the stars. But we should have *such* a measurable standard change as would be invariable; or we should have several standards which, set off against each other, measure each other perfectly. This, by experi-

* Physically we should say that the occurrences A-B and P-Q take place in exactly similar and equally isolated systems.

ence, is not possible; so the next best thing to do is to select one which is the least influenced by external changes or relative changes as our standard e.g. a chronometer; and better still to strike an average. Time is the medium for the measure of change. Suppose the system X sets off this change by which we measure Time. Strictly speaking it would be a system by which a linear change in it is measured by itself and in such a way as to satisfy all observers. (The measurement of Time would be reduced to linear measurements.) It would appear that this is not possible, hence that there is no such thing as Absolute Time; or if there is, we cannot measure it.

12.1 It is the aim of the Scientists to do away with all subjective influences whatever. Each observer has a Time, a notion of simultaneity, a means of defining the equality of Time-intervals, and hence a measurable Time. But to do away with his own part in all this he would consider the instant not of his mental impression but of the stage in the thing observed in itself; strictly speaking he would consider the instant of the mental impression he would have received had he been at the location of the stage itself i.e., therefore, irrespective of the relative location of himself and the thing observed. In this way he would attempt to render his notion of the instant an objective one. He would therefore allow by calculation for the Time-interval between the instant of an event in itself and the instant of his mental impressions, and he would put in order according to priority and subsequence all such objective instants, and measure the equality of Time-intervals by means of these objective instants. In other words he would attempt to construct Time for an abstract Mr. X, considering Mr. X as existing at all instants and in every location. But even such an attempt proves to be abortive; A's Mr. X and B's Mr. Y would themselves not agree as to *Simultaneity* or to equality of Time-intervals. What A would consider as simultaneous for Mr. X, Mr. B may not consider as such for his abstract observer Mr. Y. In other words A and B cannot come to an objective definition of Simultaneity of extrinsic events. This failure to find a suitable definition of simultaneity, on which depends also the definition of the equality of Time-intervals, would show that the concept of Time so far developed in Physics is not absolute, but only *relative*.

A METHOD FOR CONSTRUCTING MAGIC SQUARES OF THE CONCENTRIC TYPE HAVING AN EVEN NUMBER OF CELLS IN EACH ROW

BY

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1. If we remove the outermost belt of cells of a magic square of the concentric type, we will get a magic square with two cells less in each row than in the original one. If the original magic square consists of an even number of cells per row, this process of reduction can be continued until we come to a magic square of 4 cells per row. No further reduction is possible, because we cannot construct a magic square with 2 cells per row. This 4-cell-square,* got as above, by reduction, may be called the "Basic Magic Square".

2. The method, we give below, consists of, first filling in numbers in the outermost belt, then in the one next to it and so on until we are left with a 4-cell-square inside, which can be filled up by any one of the methods† of constructing a 4-cell-square. The method for filling up the outermost belt of cells of an even-cell-square is given below in § 4 and 5.

3. Before proceeding further, we define certain terms and give the meanings of some symbols we have used below. In an n-cell magic square filled up with the first n^2 natural numbers the sum of all the numbers is clearly $\frac{n^2(n^2+1)}{2}$. So the sum of the numbers in any particular column, row or diagonal, is $\frac{n(n^2+1)}{2}$. This we denote by S_n. Next the smallest and largest numbers in the magic square are 1 and n^2. Their sum n^2+1 is denoted by K_n. Thus $K_n = S_n/\frac{1}{2}n$.

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* For the sake of brevity, we call, hereafter a magic square with n cells per row as an n-cell-square.

† One of the methods for constructing a 4-cell square, is as follows:—Fill up the nos. from 1 to 16 in a 4-cell-square. Commencing in the left cell of the topmost row and proceeding to second row after the first is filled up and so on. Then write another 4-cell square and in it copy out the nos. in the two diagonals of the first square, unchanged. Interchange the first nos. of the second and third rows with the last nos. of the third and the second respectively. Interchange the second and third nos. of the top row with the third and second of the bottom row respectively. The resulting magic square is given in the margin. Instead of using the numbers 1—16 we can use any 16 consecutive numbers when necessary.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Next in an n-cell-square the result got by subtracting any number less than K_n from K_n is called the "complement" of the number. Thus in an n-cell square, the complement of x is $K_n - x$.

4. To fill up the outermost belt of a 4-n-cell square.

The outermost belt of a $4n$-cell-square consists of $(4n)^2 - (4n-2)^2$ or $16n - 4$ cells. These cells have to be filled by the first $8n-2$ natural numbers and their complements (got by subtracting from K_{4n}). This leaves the $(4n-2)^2$ numbers commencing from $8n-1$ and ending with $16n^2 - 8n + 2$ to be filled in the $(4n-2)$-cell-square which is inside this belt.

Now let us write the numbers 1 to $4n$ in two rows so that there are an equal number of them in each row, as given below :—

A	$1, 2, 3, 4, \ldots\ldots, n,$	$n+1, n+2, \ldots\ldots, 2n,$	B
C	$2n+1, 2n+2, 2n+3, \ldots\ldots, 3n.$	$3n+1, 3n+2, \ldots\ldots, 4n.$	D

They are divided into four equal groups A, B, C and D as shown above. Of the numbers belonging to groups A and D, the number 1 is placed in the topmost cell of the left column of the belt and $4n$ in the bottommost cell. The other numbers may be placed anywhere in the column. These numbers are all in the beginning of certain rows. The numbers in groups C and B are put in the right column of the belt so that they become the end numbers of rows in which there is no number in the beginning. Now we have put $2n$ numbers in each column so that every row has one and only one number and that, at the beginning or end. The sums of the numbers put in each of these two columns are equal, the sums of the numbers in the first column being $\frac{1}{2}n\ (n+1) + \frac{1}{2}n(7n+1) = n(4n+1)$ and the sum of the numbers in the last column $2n\ (4n+1)/2 = n(4n+1)$.

We next take up the top and the bottom rows. The first cells in the two rows are already filled up and we do not fill up the last cells of these rows just now. So we are left with the middle $4n-2$ cells of both the top and bottom rows. Now let us write 1, $4n$ (The numbers already placed, one in each of the two rows) and the numbers $4n+1$ to $8n-2$ in two rows consisting of equal number of terms thus :—

$$1,\ 4n,\ 4n+1, \ldots\ldots, 6n-2,$$
$$6n-1,\ 6n,\ 6n+1, \ldots\ldots, 8n-2.$$

These numbers are divided into two equal groups, so that, the first number of the first line (1), the fifth number from the beginning of the first line $(4n+3)$, and the $2n-2$ nos. in order, which have the

third and fourth terms of the second line ($6n+1$ and $6n+2$) as middle terms, form one group and the rest the second group.

The numbers of the first group, excepting 1 (which is already placed) are put in any $2n-1$ cells of the $4n-2$ cells of the top row (mentioned above). The numbers of the second group (excepting $4n$) are placed in the $2n-1$ cells of the bottom row, which are exactly below the unoccupied cells of the top row. The sums of the numbers put in each of these two rows are equal,

the sum of the numbers in the top row being

$$1+(4n+3)+\tfrac{1}{2}(2n-2)\ (12n+3)=4n+4+(n-1)\ (12n+3)=12n^2-5n+1,$$

and the sum of the numbers in the bottom row

$$3\ (4n+1)+\{\ (4n+4)+\ \ldots+(5n+2)\ \}+\{\ (7n+1)+\ \ldots+(8n-2)\ \}$$

$$=12n+3+\tfrac{1}{2}\ (n-1)\ (9n+6)+\tfrac{1}{2}\ (n-2)\ (15n-1)=12n^2-5n+1.$$

Now half of the cells in this belt are filled up. The other half is to be filled up with the complements of the numbers already put in. The complements‡ of 1 and $4n$ are put in the other end of the diagonals in which they are situated. The complements of other numbers are put in the other end of the row or column in which they are placed. Now the outermost belt is filled up.

The sum of the numbers put in first in the top row is equal to that of numbers put in the bottom row, each being equal to $n(4n+1)$. Hence the sums of the complements put in the two rows are equal. So the sums of the numbers put in the two rows are equal. If we take both these rows together they contain some $4n$ numbers and their complements. Hence the sum of the numbers in the two rows is $4n\ K_{4n}$. So the sum of the numbers in each of these two rows is $2n\ K_{4n}=S_{4n}$ as it should be. Similarly the totals of the numbers put in the first column and of the numbers in the last column must each be equal to S_{4n}. Each of the remaining columns and rows as well as the two diagonals contain only two numbers whose sum is K_{4n}.

5. To fill in the outermost belt of a $(4n+2)$ cell-square.**

The first $4n+2$ natural numbers are written in two rows of an equal number of terms; but in place of the first number of the second row $(2n+2)$, the number next to the last number in the row viz. $4n+3$ is written. The numbers are then divided into 4 groups as below :—

‡ As we are dealing with numbers in a $4n$-cell-square, the complement of a number is got by subtracting it from K_{4n}.

** In this article $n\neq 0$, for obvious reasons.

A	1, 2, 3, 4, ..., n,	$n+1$, $n+2$, ..., $2n+1$,	B
C	$4n+3$, $2n+3$, $2n+4$, ...$3n+1$,	$3n+2$, $3n+3$, ...$4n+2$.	D

[Each of groups B and D contain a term more than each of groups A and C.]

The numbers in groups A and D are placed in the cells of the left column and those in groups B and C in the right column, as already explained in the previous paragraph (§ 4). As before 1 is put in the top cell of the left column and $4n+2$ in the bottom cell of the same column.

Here also the sums of the numbers put in the two columns are equal, the sum of those in the left column being

$$\frac{n(n+1)}{2}+\frac{(n+1)(7n+4)}{2}=(n+1)(4n+2)$$

and the sum of those in the right column,

$$4n+3+\tfrac{1}{2}(n-1)(5n+4)+\tfrac{1}{2}(n+1)(3n+2)=4n^2+6n+2=(n+1)(4n+2).$$

Next we write 1, $4n+2$, (the first terms in the top and bottom rows) $2n+2$ (the term which was removed to make place for $4n+3$) and all terms from $4n+4$ to $8n+2$, in two rows consisting of an equal number of terms, as below,

$$1,\ 4n+2,\ 2n+2,\ 4n+4,\ 4n+5,\ldots\ldots\ldots\ldots 6n+1,$$
$$6n+2,\ 6n+3,\ 6n+4,\ \ldots\ldots\ldots\ldots\ldots\ldots 8n+2.$$

These terms are divided into two groups as follows:—The first group consists of 1 (the first term in first row), $4n+4$ (fourth term from the beginning of the first line), and $2n-1$ terms in order of which the third term of the second line $(6n+4)$ is the middle term. The remaining terms form the second group. The terms of these groups are used to fill up the cells of the bottom and top rows, as has already been explained.

The sums of the numbers of these two groups are equal, the sums of the numbers in the first group being

$$4n+5+\{(5n+5)+\ldots\ldots\ldots\ldots+(7n+3)\}$$
$$=4n+5+\tfrac{1}{2}(12n+8)(2n-1)=12n^2+6n+1,$$

and the sum of the numbers in the second group,

$$(4n+2)+(2n+2)+\{(4n+5)+\ldots\ldots\ldots+(5n+4)\}+\{(7n+4)+\ldots\ldots+(8n+2)\}$$
$$=6n+4+\tfrac{1}{2}(9n+9)n+\tfrac{1}{2}(15n+6)(n-1)=12n^2+6n+1.$$

The complements of the numbers placed above, are used to fill up the remaining cells of the belt as explained in § 4. (Complements are calculated here by subtracting the number from K_{4n+2}). It may easily be shown that the sum of the numbers in the two rows and in the two columns completely filled up is S_{4n+2} and that each of the remaining rows, columns and diagonals contain only 2 terms whose sum is K_{4n+2}.

6. To fill up the cells in the remaining belts.

In a $2n$-cell magic square we have to fill up $n-2$ belts before coming up to the basic magic square. For convenience of reference, let these belts be numbered from inside. Then we use the method of the last paragraph (§ 5) to fill up numbers in an odd-numbered belt and the method of § 4 to fill up numbers in an even belt.

However, there are a few modifications. When we fill up the outermost belt of a $2n$-cell square, we will have used numbers 1 to $4n-2$ and their complements. So in filling the adjacent inner belt we must use the numbers from $4n-1$ onwards instead of again beginning with 1. The resulting belt will be the same as the outermost belt of a $(2n-2)$-cell square, every number of which is increased by $4n-2$. Again when finding the complement of a no., the number must always be subtracted from K_{2n}, if we are constructing a $2n$-cell square, irrespective of the number of the belt in which the no. is placed.

After thus filling up all the belts we have to fill up the basic magic square with the numbers $2n^2-7$, $2n^2-6$, ... , $2n^2+8$ as explained in footnote § 2.

7. It has already been proved that the top and bottom rows, and the left and right columns give the required total, S_{4n} in the case of a $4n$-cell square and S_{4n+2} in a $(4n+2)$-cell-square. We may easily prove that the other columns, rows and diagonals also give the required total.

If we remove the outermost belt of a $2n$-cell-square, constructed by the above method, the magic square will be reduced into one of $2n-2$ cells per row, and the linear*-total of this magic square will be $(n-1)S_{2n}/n$. In general, if we remove the $n-r$ outermost belts of a

* The total of the numbers in any row, column, or diagonal.

$2n$-cell square, the resulting magic square will have $\frac{r}{n}S_{2n}$ for its linear-total. If we subtract $\left(\frac{r}{n}S_{2n} - S_{2r}\right) \div 2r$ i.e. $2(n^2 - r^2)$ from each of the nos. of this new magic square it will become a $2r$-cell magic square containing the natural numbers from 1 to $4r^2$.

8. A 12-cell square has been constructed by the above method and given below as an illustration.

The outermost or 4th belt.

1, 2, 3, | 4, 5, 6,
7, 8, 9, | 10, 11, 12.
} 1, 2, 3, 10, 11 and 12 are put in the left column and the rest in the right column.

1, 12, 13, 14, 15, 16.
17, 18, 19, 20, 21, 22.
} 1, 15, (fifth term from the beginning) and 18, 19, 20 and 21 (4 terms with the 3rd and 4th terms of the second line as middle terms) are put in top row and the rest in the bottom row.

The complements are got by subtraction from $K_{12} = 12^2 + 1 = 145$.

The third belt.

23, 24, | 25, 26, 27.
33, 29, | 30, 31, 32.
} 23, 24, 30, 31 and 32 are put in the left column and the rest in the right column,

23, 32, 28, 34, 35,
36, 37, 38, 39, 40.
} 23, (first term), 34 (fourth term from the beginning), 37, 38 and 39 (three terms having the third term of the second line as their middle term) are put in the top row and the rest in the bottom row.

The second belt.

41, 42, | 43, 44,
45, 46, | 47, 48.
} 41, 42, 47 and 48 are put in the left column and the rest in the right column.

41, 48, 49, 50,
51, 52, 53, 54.
} 41 (first term), 51 (fifth term from the beginning), 53 and 54 (3rd and 4th terms of the second line) are put in the top row and the rest in the bottom row.

The first belt.

55, | 56, 57,
61, | 59, 60.
} 55, 59 and 60 are put in the left column and the rest in the right column.

55, 60, 58, 62, 63, 64. } 55 (first term), 62 (4th term from the beginning), and 64 (third term in the second line) are put in the top row and the rest in the bottom row.

The other cells of the belts are filled up with the complements of the above numbers and the basic magic square with the numbers 65 – 80.

The resulting magic square is reproduced below :—

1	132	131	15	129	128	18	19	20	21	123	133
2	23	117	34	110	109	37	38	39	105	113	143
3	24	41	96	95	51	93	53	54	97	121	142
141	120	42	55	87	62	82	64	85	103	25	4
140	119	102	89	65	79	78	68	56	43	26	5
139	118	101	88	76	70	71	73	57	44	27	6
138	112	100	84	72	74	75	69	61	45	33	7
137	116	99	59	77	67	66	80	86	46	29	8
136	30	47	60	58	83	63	81	90	98	115	9
10	31	48	49	50	94	52	92	91	104	114	135
11	32	28	111	35	36	108	107	106	40	122	134
12	13	14	130	16	17	127	126	125	124	22	144

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SOME COMPARISONS OF THE GERMAN SCHOOL SYSTEM WITH OURS

BY

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The German educational system is based on the *Hoch Schule*, the High School, where boys and girls from 6 to 19 are educated. There are no Colleges in Germany. The Hoch Schule teaches up to B.A. and B.Sc. standards of our University in every subject. Every scientific subject is compulsory, including Physics, Chemistry, Botany, Zoology, Mineralogy and now-a-days Race Physiology. There are museums and laboratories attached to the Hoch Schule on the different scientific subjects.

One obvious advantage of this system is that the boys and girls get early initiated into the mysteries of science and the nation grows up with a scientific bent of mind. In our schools it is extremely difficult to introduce science teaching to any appreciable extent as they cannot pay for suitable scientific equipments and accommodations as well as for suitable teachers. One may be surprised to hear that the average pay of a German teacher in a Hoch Schule is Rs. 400 a month.

It is hopeless in this country where teachers are so poorly paid and the Schools are indifferently housed to expect anything like the German efficiency in teaching, but something might be done for the advancement of scientific teaching in important towns, a good college and a good school separately exist by associating, the higher classes of the school with the college department in the matter of scientific teaching.

It might be also desirable for the furtherance of scientific education in India to introduce in the College Curriculum for the B. Sc. degree a much larger list of compulsory scientific subjects and perhaps to abolish the Intermediate examination.

The German students are extremely assiduous in their studies, Classes are held from 8 to 11 in the morning and then again from 2 to 5 in the evening. There is an interim of 10 to 20 minutes between two lessons when the students for the most part go out of

the class to breathe fresh air. There is also a period allotted for open air drill with posturing of limbs.

I do not know how far this division of work in the school into morning and afternoon classes is feasible in India. In Germany there are no boarding schools to my knowledge, except the one I saw at Potsdam, exclusively for boys established by the Hitlerian Government on political principles. In German Schools the pupils generally go home for lunch after morning classes and come back to school again in the afternoon thoroughly refreshed. The absence of this full lunch in Indian Schools affects the health and working capacity of the students to my thinking.

In Germany there are plenty of Schools and the students do not live far away from the school. If the number of schools in India could be suitably multiplied the division of school work into morning and evening classes might be more feasible. In boarding schools this division is easy.

The Hitlerian School at Potsdam has a regulation that boys should take a nap between 2 to 3 p. m. If our boys took a nap after lunch and went to school again, it would be quite hygienic, more specially in this hot country, from the German Scientific point of view.

In the German schools many other things are encouraged, for example singing and swimming. Most Schools that are situated in the heart of a busy city have a swimming bath. Drawing and clay modelling are compulsory. I do not understand why singing should have been excluded from the school curriculum. Perhaps our school system mainly aims at qualifying our boys for certain examinations, and subjects like swimming, singing, drawing, modelling are difficult to hold general examinations on. Special certificates however could be given for these subjects.

In Berlin, there is a school for Babies, the Froebel-Christobel institu ion, which has two departments one for babies from six months to two years and the other for those above two years. The first department is not open to visitors but the other department is open to visitors once a week between specified hours. There is a special department for mentally feeble babies. These latter are mixed up with some mentally strong babies to help in countering their feeble-mindedness. The peg on which the feeble-minded baby hangs his or her coat bears a distinctive sign of a bird or beast for easy

discrimination. A large part of the teachers of the institution are ladies who get admitted into the institution in order to learn the art of bringing up babies. These young ladies later get employment in good families where their qualifications are valued. I may add that co-education is not favoured by the Hitlerian Government. In Berlin the Schools for boys and girls are sharply separated with separate curricula. In the larger towns co-education still largely exists in the High Schools as a High School in Germany is an expensive matter. In India where the High Schools do not teach up to College Standard the schools for boys and girls are generally sharply separated, but a separation of curricula is also desirable.

The educational authorities of Hamburg and the Minister of Education of the Hitlerian Government were good enough to grant me permission to visit the Schools in Hamburg and Berlin and attend whatever classes I chose. I took advantage of this kind permission to visit the best Schools in and about Hamburg and Berlin during my short stay in Germany and sit in the classes which interested me most. I regret I could not find time to fully utilize this kind permission. Nevertheless I was able to see some of the best German Schools personally and sit at the classes, a happy experience which has fallen to the lot of very few Indians.

GLEANINGS

50. A Mathematician who is not also something of a poet will never be a complete mathematician.

WEIERSTRASS.

51. The notion of infinity is our greatest friend ; it is also the greatest enemy of our peace of mind . . . . Weierstrass taught us to believe that we had at last thoroughly tamed and domesticated this unruly element. Such, however, is not the case ; it has broken loose again. Hilbert and Brouwer have set out to tame it once more. For how long ? We wonder.

JAMES PIERPONT.

SOME REMARKS ON BELTRAMI'S ASSOCIATED RULED SURFACE WITH PARALLEL GENERATORS

BY

RAM BEHARI, *University of Delhi.*

1. The object of this paper is to point out some instances of ruled surfaces which though applicable to a given ruled surface are not mutually deformable to it, and also to verify that for Beltrami's Associated Ruled Surface with parallel generators, L, M, N, the fundamental magnitudes of the second order have opposite signs.

2. Let the equations of a ruled surface be

$$x = p + lu, \quad y = q + mu, \quad z = r + nu,$$

where $p, q, r; l, m, n$ are functions of v, the arc of the base curve.

The square of the linear element is given by

$$ds^2 = du^2 + 2\mathrm{D}du\,dv + (\mathrm{A}u^2 + 2\mathrm{B}u + 1)dv^2,$$

where $\sum l^2 = 1$; $\sum p'^2 = 1$; $\sum l'^2 = \mathrm{A}$; $\sum lp' = \mathrm{D}$; $\sum l'p' = \mathrm{B}$.

Thus there are five relations between six quantities p, q, r, l, m, n which determine the ruled surface. Hence there is an infinite number of ruled surfaces having a given linear element.

Choosing $l = \sin\omega\cos\psi$, $m = \sin\omega\sin\psi$, $n = \cos\omega$ so that the relation $\sum l^2 = 1$ is satisfied, and choosing $\psi = \int\sqrt{\mathrm{A} - \omega'^2}dv / \sin\omega$ where ω is an arbitrary function of v, so that the relation $\sum l'^2 = \mathrm{A}$ is also satisfied, and then solving

$$p'l + q'm + r'n = \mathrm{D}, \qquad p'l' + q'm' + r'n' = \mathrm{B},$$

and

$$p'(mn' - m'n) + q'(nl' - n'l) + r'(lm' - l'm) \equiv -\mathrm{VM},$$

where

$$\mathrm{V} = \sqrt{\mathrm{EG} - \mathrm{F}^2}, = -(\mathrm{A} - \mathrm{AD}^2 - \mathrm{B}^2)^{\frac{1}{2}} = \pm\mathrm{A}\beta,$$

E, F, G, being the fundamental magnitudes of the first order, and β the parameter of distribution, we get

$$p' = \mathrm{D}l + \frac{\mathrm{B}}{\mathrm{A}}l' \pm \beta(mn' - m'n),$$

$$q' = \mathrm{D}m + \frac{\mathrm{B}}{\mathrm{A}}m' \pm \beta(nl' - n'l),$$

$$r' = \mathrm{D}n + \frac{\mathrm{B}}{\mathrm{A}}n' \pm \beta(lm' - l'm),$$

which give two values for each of the quantities p', q', r', corresponding to any of the infinite sets of values of l, m, n. Hence we get the

THEOREM OF BELTRAMI:

> Given a ruled surface there is another (and different) ruled surface applicable in such a way that the corresponding generators are parallel and in the same sense.*

Thus Beltrami's associated ruled surface with parallel generators differs from the original ruled surface in having an opposite sign for β, the parameter of distribution.**

But since by Chasle's Theorem † $\tan\phi = u/\beta$ where u is the distance of a point on the generator measured from the central point of that generator and ϕ is the angle which the tangent plane at the point makes with the central plane, it follows that the tangent plane rotates in opposite directions as the point of contact moves along the two corresponding generators on the respective surfaces. Hence the two ruled surfaces of Beltrami's Theorem, though applicable, are not mutually deformable. Since

$$\beta = -\begin{vmatrix} p' & q' & r' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \Big/ \sum l'^2,$$

it is obvious that β changes sign whenever all the elements of any one column or all the elements of all the three columns change signs. Thus, some other instances of ruled surfaces applicable but not mutually deformable to the ruled surface $x = p + lu$, $y = q + mu$, $z = r + nu$ are:—

$$x = -(p + lu), \quad y = q + mu, \quad z = r + nu;$$
$$x = p + lu, \quad y = -(q + mu), \quad z = r + nu;$$
$$x = p + lu, \quad y = q + mu, \quad z = -(r + nu);$$
$$x = -(p + lu), \quad y = -(q + mu), \quad z = -(r + nu).$$

More generally the two surfaces

$$x = f(u, v), \; y = g(u, v), \; z = h(u, v);$$

and $x' = -f(u, v)$, $y' = g(u, v)$, $z' = h(u, v)$ have the same ds^2, since $dx^2 = dx'^2$, $dy^2 = dy'^2$, $dz^2 = dz'^2$, but are not in general deformable. They are reflections of each other in the x-plane.

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* See Forsyth, *Differential Geometry* p. 388.

** Eisenhart, *Diff. Geometry* p. 345.

† See Wilczynski's *Projective Diff. Geometry* p. 136.

We have, where accented letters refer to the second surface, $E' = E$, $F' = F$, $G' = G$, $V' = V$; $L' = -L$, $M' = -M$, $N' = -N$,

$$\text{since} \quad L = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_{11} & y_{11} & z_{11} \end{vmatrix} /V, \quad M = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_{12} & y_{12} & z_{12} \end{vmatrix} /V, \quad N = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_{22} & y_{22} & z_{22} \end{vmatrix} /V$$

change sign if the sign of one or of all three of the functions $x, y, z,$ is changed.

Since E, F, G, L, M, N determine the surface, except in position, by Bonnet's Theorem,* it follows that the surface determined by E, F, G, L, M, N, and the surface determined by E, F, G, $-L$, $-M$, $-N$ are congruent after one of them has been subjected to a reflexion in a plane.

Now, consider surfaces with a given linear element, *i.e.*, with the same E, F, G.

We know that L, M, N of each surface must satisfy the Gauss characteristic equation †

$$LN - M^2 = -\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11}) + (E, F, G)(\Gamma', \Delta')^2 - (E, F, G)(\Gamma'', \Delta'') \quad \ldots(1)$$

and the two Mainardi-Codazzi relations †

$$L_2 + \Gamma M + \Delta N = M_1 + \Gamma' L + \Delta' M \quad \ldots(2)$$

$$M_2 + \Gamma' M + \Delta' N = N_1 + \Gamma'' L + \Delta'' M \quad \ldots(3)$$

These conditions are necessary for the determination of L, M, N, being the conditions of integrability. They are also sufficient because when they are satisfied then by Bonnet's Theorem for every one solution of L, M, N, a unique surface is determined save as to its position and orientation in space.

Equations (1), (2), (3) show that if L, M, N is one solution then $-L$, $-M$, $-N$ is also a solution.

The corresponding surfaces given by E, F, G, L, M, N; and E, F, G, $-L$, $-M$, $-N$, by Bonnet's Theorem are distinct.

Also since the perpendicular distance of the tangent plane to the surface at a point (u, v) from a neighbouring point $(u + du, v + dv)$ is, to the second order of small quantities, $\frac{1}{2}(L du^2 + 2 M du\, dv + N dv^2)$,** it follows that the perpendiculars on the tangent planes to the two

* Forsyth, *Diff. Geom.* p. 50.

† Forsyth, *Diff. Geom.* pp. 46, 48.

** See Eisenhart. *Diff. Geom.* p. 114.

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surfaces, (viz., corresponding to the solutions L, M, N; and $-L$, $-M$, $-N$) at two corresponding points from two corresponding neighbouring points have opposite signs, and hence the tangent plane rotates in opposite directions as the point of contact moves along corresponding curves on the two surfaces. The two surfaces are, therefore, not mutually deformable, though they are applicable. Hence we get the general result that :—

All surfaces applicable to a given surface are given by (1), (2), (3), and corresponding to every solution L, M, N of the equations (1), (2), and (3), there is another solution —L, —M, —N which gives a surface applicable but not mutually deformable to the given surface. *

3. We shall now verify that for Beltrami's associated ruled surface with parallel generators, L, M, N have opposite signs. In the case of the ruled surface given by

$$ds^2 = du^2 + 2\mathrm{D}du dv + (\mathrm{A}u^2 + 2\mathrm{B}u + 1)dv^2,$$

we have

$\mathrm{E} = 1$, $\mathrm{F} = \mathrm{D}$, $\mathrm{G} = \mathrm{A}u^2 + 2\mathrm{B}u + 1$, so that $\mathrm{V}^2 = \mathrm{G} - \mathrm{F}^2$; $\mathrm{E}_1 = \mathrm{E}_2 = 0$; $\mathrm{F}_1 = 0$.

The Gauss' characteristic equation (1) which gives the value for M reduces to

$$\mathrm{VM} = \sqrt{(\mathrm{A} - \mathrm{AD}^2 - \mathrm{B}^2)} \quad = \pm \mathrm{A}\beta \qquad \dots (4)$$

where β is the parameter of distribution.

The Mainardi–Codazzi relation (2) becomes $\mathrm{M}_1 + \triangle'\mathrm{M} = 0$,

i.e., $\mathrm{M}_1 + \dfrac{\mathrm{G}_1}{2\mathrm{V}^2}\cdot\mathrm{M} = 0$, since $\mathrm{V}^2\triangle' = n'\mathrm{E} - m'\mathrm{F} = \frac{1}{2}\mathrm{G}_1$;

$$\therefore \quad \mathrm{M}_1 + \frac{\mathrm{VV}_1}{\mathrm{V}^2}\mathrm{M} = 0, \text{ since } \mathrm{V}^2 = \mathrm{G} - \mathrm{F}^2 \text{ gives } 2\mathrm{VV}_1 = \mathrm{G}, \text{ or } \mathrm{M}_1\mathrm{V} + \mathrm{V}_1\mathrm{M} = 0,$$

$$\text{or } \frac{\partial}{\partial u}(\mathrm{MV}) = 0, \text{ or } \mathrm{MV} = f(v) \qquad \dots (5)$$

where f is an arbitrary function.

Comparing this with (4) we see that this arbitrary function $f(v)$ is $\pm \mathrm{A}\beta$.

When β changes sign, M also changes sign. Also the Mainardi-Codazzi relation (3) is satisfied when M and N both change sign.

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* Cf. B.M. Sen, "On the applicability and deformation of surfaces", *Quarterly Journal* 1927. p. 199; also 'A note on the deformation of surfaces', *Bull. Cal. Math. Soc.* 1919 pp. 213–219.

Thus we see that the peculiarity of Beltrami's associated ruled surface lies in the fact that in this case the general equations (1), (2), (3) are capable of easy solutions.

N.B.—That L, M, N have opposite signs for Beltrami's associated ruled surface can also be seen easily as follows:—

Taking an orthogonal trajectory of the generators as the base curve, we know that $L = 0$, $M = \frac{A\beta}{V}$,

$$N = -\frac{1}{VA\beta}\left\{u^2\left(At - \tfrac{1}{2}BA'\right) + u\left(2Bt - \tfrac{1}{2}A' - BB'\right) + \left(t - B'\right)\right\},$$

where $t = \sum l''p'$.* Hence L, M, N change sign with β.

* See Eisenhart, *Diff. Geom.* p. 249.

SOME REMARKS ABOUT RELATIVITY.

52. A friend of mine was once at a dinner-party and the lady next to him said: "Professor, do tell me in a few words what this Theory of Relativity really is." He replied: "Of course I will—provided you will let me tell you this little story first. I was going for a walk with a French friend and we got thirsty. By and by we came to a farm and I said: 'Let us buy a glass of milk here'. 'What's milk?' 'Oh, you dont know what milk is? It is the white liquid that—' 'What's white?' 'White? You dont know what that is either? Well, the swan ——' 'What's swan?' 'Swan, the big bird with the bent neck.' 'What's bent?' 'Bent? Good heavens, dont you know that? Here look at my arm: when I put it so, it is bent!' 'Oh, that is bent, is it? Now I know what milk is!'"

MAX BORN in *The Restless Universe.*

NOTES AND DISCUSSIONS

A Completion of the Fermat-Euler Theorem

I. Fermat's Theorem states that if n be a prime number and a be any number prime to n, then

$$a^{n-1} \equiv 1 \pmod{n} \quad \ldots (1)$$

This means that in the series

$$a,\ a^2,\ a^3,\ \ldots\ \ldots \quad \ldots (2)$$

the residues of the successive terms recur periodically, the number of residues in each period being $(n-1)$ or a factor of $(n-1)$.

Euler proved that if a and n are two numbers prime to each other and $\phi(n)$ the number of integers, including unity, which are less than n and prime to n, then

$$a^{\phi(n)} \equiv 1 \pmod{n} \quad \ldots (3)$$

This means that when a is prime to n, the number of residues in each period of the series (2) is $\phi(n)$ or a factor thereof.

These theorems do not give information regarding the behaviour of the series (2) (mod n) when a is not prime to n. The object of this note is to show that there is an initial non-recurring part and a recurring part in the residues of the successive terms of the above series, and to find out the length of the acyclic and the cyclic parts.

II. Since there are only a finite number of distinct residues (mod n), there must be a first term in the series (2) which leaves the same residue as a subsequent term, that is, there is a least value of say r, and a least value of k for which we must have

$$a^r \equiv a^{r+k} \pmod{n}.$$

This implies $a^{r+l} \equiv a^{r+k+l} \pmod{n}$, or in other words that the period of the series begins from a^r, and is of length k.

The following theorem determines the length, $(r-1)$, of the acyclic part and the period k as functions of a and n.

THEOREM: *Let the acyclic part contain $(r-1)$ terms and let k be the period. Then*

(1) *r is the least number such that the greatest common divisor, g, of a^r and n is a block factor* of n;*

(2) *k is the exponent to which a appertains (mod n/g).*

Proof: Let g_1 be the greatest common divisor of a^s and n.

Suppose $$a^{\lambda+s} \equiv a^s \pmod{n}.$$

Then we have
$$a^s(a^\lambda - 1) \equiv 0 \pmod{n},$$
$$\therefore \quad a^s(a^\lambda-1)/g_1 \equiv 0 \pmod{n/g_1}.$$

Since a^s/g_1 is prime to n/g_1, this gives
$$a^\lambda - 1 \equiv 0 \pmod{n/g_1}.$$

Now, if $s < r$, g_1, and therefore a, has a factor in common with n/g_1 and therefore the above congruence is impossible. If, on the other hand, $s=r$, then from the definition of r, g_1 as well as a^s/g_1 are relatively prime to n/g_1; therefore a is prime to n/g_1 and the above congruence is possible if λ is the exponent to which a belongs (mod n/g_1), that is, if $\lambda = k$.

The following special cases of the Theorem are noteworthy.

1. The theorem implies Euler's result (2). For, if a is prime to n, then $r=0$ and we have
$$a^k \equiv 1 \pmod{n}.$$

2. If the g. c. d. of a and n is a block factor of n, then $r=1$ and we have
$$a^{k+1} \equiv a \pmod{n}$$
so that there is no acyclic part in this case.

3. When $n = n_1 n_2$ where n_1 and n_2 are relatively prime, then we can find a so that, if t is a factor of n_1 which contains all the prime factors of n_1)
$$a \equiv 0 \pmod{t}; \quad a \equiv 1 \pmod{n_2}.$$

In this case it is clear that $g = n_1$, and k, the exponent to which a appertains (mod $n/g = n_2$), is 1.

Hence we have
$$a^r \equiv a^{r+1} \equiv a^{r+2} \equiv \cdots \pmod{n}.$$

* A factor d of n is called a block factor, if it is relatively prime to n/d.

III. We shall now consider an application of the theorem. Let $d_1 d_2 \ldots d_s$ be the last s digits of a number a, where $d_s = 5$; so that,

$$a = d_1 d_2 \cdots d_s = 5^t \cdot \mathrm{A} \pmod{10^s} \text{ where A is prime to 5.}$$

We shall now determine the last s digits of powers of this number, in other words, the acyclic part and the period of the series

$$a, a^2, a^3, \ldots \pmod{10^s}.$$

Since a ends in 5, it is prime to 2. Therefore we have from the main theorem

r = the integer equal to or next $> s/t$,
k = the exponent to which $(\mathrm{A} \times 5^t)$ appertains $\pmod{2^s}$.

From known* results we have, for $s > 2$, $\mathrm{A} = \pm 5^k \pmod{2^s}$, and, if the highest power of 2 contained in $(k+t)$ be λ, then the exponent to which $(\pm 5^k \cdot 5^t)$ belongs $\pmod{2^s}$ is equal to $2^{s-2-\lambda}$.

When $s = 2$, $a = d_1 5 \pmod{10^2}$, and we have from the main theorem

$r = 2$ or 1 according as 5 or 5^2 divides $d_1 5$,
k = the exponent to which $d_1 5$ appertains $\pmod{2^2}$.

If d_1 is even, $d_1 5 \equiv 1 \pmod{2^2}$ so that $k = 1$.

If, on the other hand, d_1 is odd, it is known† that

$$(d_1 5)^2 = (d_1^2 + d_1)\, 25 \equiv 1 \pmod{2^2} \text{ so that in this case } k = 2.$$

I am thankful to Dr. R. Vaidyanathaswamy for his helpful suggestions.

Madras University. K. SUBBA RAO.

On the Lines of Curvature of some Developables.

THEOREM I.—*The developable surfaces whose lines of curvature are all plane curves are those generated by the tangents of a helix, together with the cylinder and the right circular cone.***

The lines of curvature of any cylinder are plane curves viz. the generators and the cross-sections of the cylinder. The lines of curvature (other than the generators) of a general cone are curves cutting the generators orthogonally and hence lie on concentric spheres with the vertex of the cone as centre. If these curves are to be plane

* Landan, "*Vorlesungen uber Zahlentheorie*", Band I, Theorems 125 and 126.
† K. Subba Rao, "Multiplication of numbers ending in 5", *Math. Student*, Vol. 12, p 72.
** Also, see Wilkinson *Journ. Ind. Math. Soc.* Vol. IV, p. 209.

curves, they are circles, being plane sections of spheres. If A be the centre of one of these circles, and O the vertex of the cone i.e. the centre of the sphere, AO is perpendicular to the plane of the curve. If P be a variable point on the circle, then in the triangle OAP, the lengths OA, AP and the angle OAP are constant. Hence $A\hat{O}P$ is constant. The cone is thus right circular.

For a general developable, the lines of curvature are the generators and the involutes of the edge of regression. But if the involutes are plane curves, it is well-known that any skew evolute is a helix.

THEOREM II.—*The only surface all of whose lines of curvature are geodesics is the cylinder.*†

If the lines of curvature are all geodesics, the surface will possess two sets of geodesics which are orthogonal trajectories of each other and is therefore a developable. Also, if a curve is a geodesic as well as a line of curvature, it is a plane curve. Hence the developable, if it is not a cone or cylinder has, by Theorem I, a helix for its edge of regression. But in order that a non-rectilinear line of curvature should also be a geodesic, the tangents to the helix should be binormals of the curve. This is impossible for a developable having a proper skew edge of regression, since the binormals of a plane curve are all parallel to a fixed direction. Hence the result follows after noting that the cross-sections of a right circular cone are not geodesics, while any cross-section of any cylinder is evidently a geodesic.

[I regret an error in the first part of question 1554 solved in J. I. M. S. Vol XIX Part II. pp. 246 – 247. Any developable could be expressed with the arc-element in that form, or in the form $ds^2 = \lambda \; dp \; dq$ where λ is a constant. The argument employed is correct only if L/N is a function of p alone. The surface is then a cylinder by Theorem II above. This is illustrated by the example in the second part of the question].

THEOREM III—*The non-rectilinear lines of curvature of the developable formed by the tangents of (a) a circular helix, (b) an equiangular helix are respectively (a) involutes of a circle, (b) equiangular spirals.*

† This is an immediate inference from Ex. 22 p. 123 of Weatherburn: *Differential Geometry I.* His method is based on considerations connected with geodesic torsion. Vide p. 104 Ex. 1 and p. 122 Ex. 22.

(*a*) expressed in other words is given as Ex. 2. § 10, Weatherburn *loc. cit.* Both the results are obtained from the equations

$$\rho_1{}^2 = \frac{(c-s)^2\ \sigma^2}{\rho^2+\sigma^2}\ ; \qquad \frac{ds_1}{ds} = \frac{c-s}{\rho}$$

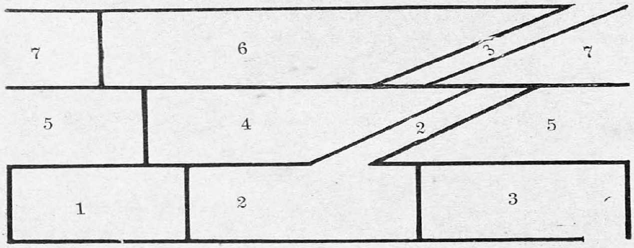
where ρ, σ, s refer to the evolute and ρ_1, s_1 to the involute. To prove (*b*), we note that $\rho = a\ (s-c')$ for an equiangular helix i.e. for a helix the cross-section of whose rectifying developable is an equi-angular spiral. Hence, $\frac{ds_1}{ds} = \frac{c-s}{a(s-c')}$. Obviously we must have $c=c'$, since otherwise s_1 would be infinite when $s=c'$. Hence $s_1+b\ = -s/a$ and $\rho_1 = l\ (c-s)$.

Hence $\rho_1 = \mathrm{A}\ (s_1 - \mathrm{B})$ where b, l, A, B are constants. The plane curve which has this property is the equi-angular spiral.

Bangalore. C. N. SRINIVASIENGAR.

The division of the surface of the anchor-ring into seven neighbouring regions.

The maximum number of regions lying on the surface of a ring which are such that each region adjoins every other region is known to be seven. The following is a simple way of dividing it into seven regions possessing the preceding property and it appears to be simpler than the previous ones [see e. g. Saint Veuant: *Les reseax—Hilbert—Anschainleche Geometry.*] The anchor-ring can easily be cut up into a rectangle whose opposite sides are to be considered as identical. The figure explains the construction of the seven regions.



K. S. VENKATACHALA IYENGAR.

A result in the Geometry of the Triangle

The problem of "drawing from the vertices of a triangle to the opposite sides three segments of the same length λ, all being on one point" is solved in page 216 of Morley and Morley's: "Inversive Geometry". There are two points satisfying the conditions of the problem and they are given by the equation

$$\sqrt{x-x_1}+\sqrt{x-x_2}+\sqrt{x-x_3}=0 \qquad \dots (1)$$

where $x\ x_1\ x_2\ x_3$ are complex numbers representing the point in question and the vertices of the triangle. The two points are however not identified. I find that they are the foci of the Steiner Circumconic of the triangle.

The equation (1) gives * the foci of the curve whose areal tangential equation is

$$\sqrt{l}+\sqrt{m}+\sqrt{n}=0 \qquad \dots (2)$$

This is the tangential equation of a circumconic whose centre is the centroid of the triangle.

The common value of the segment λ is $3a/2$, where a is the semi-major axis of the ellipse.

N. DURAIRAJAN.

ASTRONOMICAL NOTES FOR MAY & JUNE 1937

Sun.—During the month of May the Sun will be moving in a North-easterly direction from the central part of the Aries coustellation and after passing the central part of Taurus by the end of May, reaches the central part of Gemini by the end of June 1937. The position of the Sun will be as follows on the 1st of May and June for Trivandrum Mean noon.

1st May	R. A. 2 h 32 m	Decl. 14° 58′ North
1st June	2 h 25 m	22° 1′ North

Moon.—The following phenomena of the Moon will occur during the month of May and June.

* Please see my paper on "Foci in Complex Geometry" *Jour. Ind. Math. Soc.* Vol. 17, p. 224.

	MAY.	JUNE.
Last Quarter	3rd at 11-45 P. M.	2nd at 10-32 A. M.
New Moon	10th at 6-26 P. M.	9th at 1-51 A. M.
First Quarter	17th at 11-57 A. M.	16th at 0-10 A. M.
Full Moon	25th at 10-46 P. M.	24th at 4-8 A. M.
Apogee	24th at 6-8 P. M.	21st at 1-8 A. M.
Perigee	10th at 11-8 P. M.	8th at 8-8 A. M.

Mercury—will be in the Aries constellation during May and passes to the Taurus by the 7th of June and will remain there throughout the month. The planet will be an evening star till May 11th and on this date of inferior conjunction with the Sun the planet will partially transit across the Sun's disc, the maximum ingress being only 7/12th of the planet's disc. At Trivandrum the ingress will occur at 2-9 P.M. and egress at 2-51 P.M. (Indian Standard Time), at Madras these times are 2-14 and 2-46 P.M. (I. S. T.). By the end of May it will be a morning star rising about an hour and a half before the Sun. On May 23rd the planet will be stationary. Mercury will be at greatest elongation on 6th June and on next day it will be in conjunction with the Moon. The position of Mercury on the 1st of May and June will be as follows.

1st May	R. A. 3 h 26 m	Decl. 21° 1′ North
1st June	3 h 5 m	31° 27′ North

Venus—will be in the constellation Pisces during the beginning of May and it will be visible as a morning object rising by about 3 hours before the Sun. Venus is stationary on the 6th May and from this date its motion will be direct. It is in conjunction with the Moon on 8th May and attains its greatest brilliancy on May 24th. On 23rd May it enters Aries and remains in Aries till the end of June. During June it will be in conjunction with the Moon and Uranus on 6th and 18th June respectively. The position of Venus during the 1st of May and June are

1st May	R. A. 1 h 11 m	Decl. 11° 27′ North
1st June	1 h 47 m	9° 2′ North

Mars.—This planet will be a bright object in the constellation Scorpio throughout May being West of Antares. It will be visible during the major part of the night. It will be in opposition with the Sun on May 19th and conjunction with the Moon on 24th May. On May 28th it is nearest the earth. During the first week of June it enters Libra. On 20th June it is in conjunction with the Moon and it is stationary on the 27th June. Its positions are as follows:

1st May	R. A. 16 h 7 m	Decl. 20° 54′ South
1st June	15 h 25 m	20° 14′ South

Jupiter.—During the months of May and June, Jupiter will be in Capricornus and is visible after midnight, being overhead early in the morning. It is in conjunction with the Moon on 2nd, 29th May and 25th June. It is stationary on the 15th May. Its positions will be as follows

1st May	R. A. 19 h 56 m	Decl. 20° 55′ South
1st June	19 h 55 m	21° 1′ South

Saturn—will be in Pisces during May and June. It is a morning object rising about two and a half hours after midnight in the middle of May. Its rings are gradually widening and could be seen with the aid of small telescopes. It is in conjunction with the Moon on 7th May and 4th June. Its positions will be as follows

1st May	R. A. 0 h 6 m	Decl. 1° 38′ South
1st June	0 h 16 m	0° 38′ South

Uranus—will be very near the Sun in apparent position since the conjunction of the planet with the Sun occurred on 30th April. Its position is as follows.

1st May	R. A. 2 h 30 m	Decl. 14° 19′ North
1st June	2 h 36 m	14° 51′ North

Neptune. This planet will be in Leo and will be seen as an evening object. It is stationary on the 28th May. Its position is defined by the following co-ordinates

1st May	R.A. 11 h 12 m	Decl. 6° 23′ North
1st June	11 h 11 m	6° 27′ North

Comets. A new comet was reported to have been discovered by A. Wilk at Cracow on February 27 and later by L. C. Peltier at Delphos. It is named as Comet (Wilk-Peltier) 1937 c. It was of 7th magnitude at the time of discovery. Comet 1937 b. was discovered on 7th February by Dr. Whipple. It was of 12th magnitude.

The star α Aurigae (Capella) was found by Campbell to be a spectroscopic binary with a period of 104.022 days. The magnitude of its companion is fixed as 10.6. By further search it was confirmed by Kuiper that Capella appeared to be a quadruple system of the ξ Lyrae type.

Novae. Nova Herculis that burst in 1934 December is keeping at a steady magnitude of eight with small changes in apparent brightness. The two new stars in Aquilae are slowly getting fainter, the present magnitude being nearly 10.

In a paper in the Monthly Notices of the Royal Astronomical Society Dr. H. Jeffreys has derived the following value for the ellipticity of the earth viz., $1/296{\cdot}38 \pm 0{\cdot}51$ and a modified formula for gravitation viz.,

$$g = 978{\cdot}051\{1 + (0{\cdot}005282 \pm 0{\cdot}000006)\sin^2\phi - 0{\cdot}000007\sin^2 2\phi\}$$

He deduces the following results for the Moon :— The ellipticity of the equator is about one sixth that of the disc. The rotation period of the Moon when it solidified was about 3·5 days.

Matter in the Universe and the gravitational constant. Prof. P. A. M. Dirac has obtained a number comparable with the ratio of the electric to the gravitational force between the electron and the proton (10^{39}), by expressing the age of the universe as derived from cosmological theories (2×10^9 years) in units provided by atomic constants (e^2/mc^3). It is suggested that this ratio and also that between the mass of the universe and the mass of the proton (10^{78}) are functions of the age of the universe. From these and other considerations he infers that the amount of matter in the universe is constantly increasing, while the gravitational constant decreases.

Constitution of the earth. Dr. H. Jeffreys summarises our knowledge regarding the constitution of the earth in a lecture before the Royal Astronomical Society. The mode of transmission of the earthquake waves proves that up to a depth of 3000 K.M. there is a shell of increasing density from 2·7 to 5. The principal constituent of the earth's solid shell is olivine. This however does not extend to the surface, where there is a layer of granite under the accumulated sedementary rocks of the continents and a layer of denser rocks, presumably basic, lies under the oceans. This layer is about 10 K.M. thick. There are also two other layers of about the same thickness, but discontinuous in their density lying between this layer and the lower shell of Olivine. It is also inferred that radio activity rapidly decreases with depth.

Ionised titanium.—The existence of ionised titanium in interstellar space has been inferred by Dr. T. Dunham Jun from a study of the ultra-violet spectra of several stars.

The diameter of our Galaxy of stars.—The colours of about 68 clusters of stars studied at Mount Wilson by Stebbins and Whitford showed that the absorption of light in space and the reddening effect of great distances show non-uniformity. The colour excess shows marked correlation with space absorption as deduced from a number of extra galactic nebulae in the field, and also with galactic latitude, being redder near the galactic equator, where the absorption should be greatest. Due to this kind of absorption, the distances inferred from apparent magnitudes in clusters has to be divided by 4. The diameter of the galaxy when this correction is applied comes to 30,000 parsecs, which is the order of the diameters of some of the spiral nebulae.

Solar Corona.—It is established by W. M. Cohn that both bound and free electrons are present in the solar corona. He has obtained curves showing the polarisation of light of different wavelengths in the solar corona. The results show that coronal light is not chiefly due to the scattering of sun light by free electrons. The corona itself emits unpolarised light.

Trivandrum. H. SUBRAMANI AYAR

SOLUTIONS TO QUESTIONS

Question 1575

(B. RAMAMURTI):—ABC, A′B′C′ are two cubicovariant triads on a twisted cubic. H_1, H_2 are the Hessian points. The tangent at H_1 meets the osculating plane at H_2 in P, and the tangent at H_2 meets the osculating plane at H_1 in Q. Show that the planes ABC, A′B′C′ and the osculating planes at H_1 and H_2 form a harmonic axial pencil. Show also that the quadrics through ABC, A′B′C′ and P, Q touch at P and Q, the tangent planes being P H_1 H_2 and Q H_1 H_2.

Remarks by C. N. Srinivasiengar.

For the straightforward proof of the first part, vide: *Math. Student*, Vol. I, p. 111. We may observe the following interpretation of the result: The tangents of a twisted cubic belong to a linear complex. A twisted cubic sets up a (1, 1) correspondence between points and planes of space, since through any point there pass three osculating planes whose contact points lie in a plane through the point. Moreover, this correspondence is identical with that set up by the linear

complex of the tangents, through its polar planes and poles. Taking the cubic as $x : y : z : w = t^3 : t^2 : t : 1$, the polar plane of (x_1, y_1, z_1, w_1) with respect to this linear complex is

$$xw_1 - 3yz_1 + 3zy_1 - wx_1 = 0.$$

Hence the poles of the planes $x - w = 0$ and $x + w = 0$ are respectively $(1, 0, 0, 1)$ and $(1, 0, 0, -1)$. These points evidently separate harmonically the points H_1 and H_2, viz. $(0, 0, 0, 1)$ and $(1, 0, 0, 0)$. Thus,

the poles of the planes defined by a pair of cubicovariant triads on a twisted cubic, with respect to the linear complex of the tangents of the curve separate harmonically the Hessian points of the triads.

The second part of the question rests upon the fact that the two conics through A, B, C, P, Q and A′, B′, C′, P, Q are common to every quadric through the eight points. The quadrics therefore touch at P and Q.* Among these quadrics, there is one which contains the cubic curve. Its equation is easily found to be given by $yz = xw$. The plane $y = 0$ evidently touches this quadric at $(0, 0, 1, 0)$ since the section of the quadric by the plane $y = 0$ is a pair of lines meeting at P. Hence all quadrics of the system have $y = 0$ i.e. the plane PH_1H_2 as the tangent plane at P. Similarly QH_1H_2 is the tangent plane at Q.

We have seen that the pole of the plane ABC with respect to the complex determined by the tangents to the cubic curve lies on $H_1 H_2$. Also the polar plane of P is evidently the plane $PH_1 H_2$, since the osculating planes at H_1 and H_2 pass through P. It now easily follows that *the tangents at P and Q to the conic ABCPQ meet on $H_1 H_2$ at the pole of the plane ABC with respect to the complex, and the tangents themselves belong to the line-complex.*

The twisted cubic does not admit of simple geometrical covariant properties of sets of points on it with the same ease as the conic does. As additions to the property in Qn. 1575, the following few properties may be mentioned here:

(i) The condition that the two cubics $a_0 t^3 + 3a_1 t^2 + 3a_2 t + a_3 = 0$ and $b_0 t^3 + 3b_1 t^2 + 3b_2 t + b_3 = 0$ should be apolar is equivalent to the condition that the planes containing the two triads of points on the twisted cubic should be conjugate, in the sense that the pole of either plane with respect to the linear complex of tangents of the cubic curve lies on the other.

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* **Bell**: *Co-ordinate Geometry of Three Dimensions.* § 169.

(ii) If $f(t, t') \equiv a_0 t^4 + 4a_1 t^3 + 6a_2 t^2 + 4a_3 t + a_4 = 0$ denote a tetrad of points A, B, C, D on the cubic, the polar plane of any other point (u, u') on the cubic with respect to the tetrahedron ABCD is the same as the plane joining the triad $u \frac{\partial f}{\partial t} + u' \frac{\partial f}{\partial t'} = 0$ where u' and t' are introduced for the sake of homogeneity.

The proof is the same as that for a similar result about triads on conics. Vide Sommerville: *Analytical Conics*, p. 250.

Corollary: If A, B, C, D be four fixed points on a twisted cubic, and P a variable point on it, the locus of the pole with respect to the tangent complex, of the polar plane of P with respect to the tetrahedron ABCD is a straight line.

(iii) Two tetrads ABCD, A′B′C′D′ of points on a twisted cubic are apolar if the triad BCD is apolar to the triad in which the polar plane of A with respect to the tetrahedron A′B′C′D′ meets the cubic. Compare Sommerville *loc. cit.* p. 254.

Reference may be made to Meyer's *Apolaritat and Rationale Kurven* for further information.

Question 1604

(B. RAMAMURTI):—A variable line intersects a pair of opposite edges of a tetrahedron, inscribed in a quadric Q. Show that the locus of the double points of the involution on the line determined by the pairs of points of intersection with the pair of opposite edges and the quadric, is a quadric containing the four other edges and doubly apolar to Q.

Solution by K. Rangaswami.

An arbitary line l together with the pair of opposite edges, say, AB, CD of the tetrahedron ABCD uniquely determines a quadric Γ of which these are the generators of the same system. Let l' be the generator of this system, so that l, l'; AB, CD are harmonic pairs in this regulus of Γ. Now if γ is the quartic curve of intersection of Γ and Q, the locus of points, one on each generator g of Γ of the system opposite to l, which with the meet of g with l separates harmonically, the intersections of g with γ, is a twisted cubic (being the locus of intersection of corresponding planes in three related pencils of planes) which meets l' in two points p_1, p_2. Thus, each of the generators g_1, g_2 through p_1, p_2 of the system opposite to l meets l in a point of the required locus. And, on the arbitary line l there being only two such points, the locus is a

quadric S which is easily seen to contain the four other edges of the tetrahedron. Further the quartic of intersection of S and Q contains one inscribed generator-quadrangle of S viz., ACBD; this relation is poristic and consequently S is doubly apolar to Q.

Question 1630

(T. R. RAGHAVA SASTRI):—If two triangles are in perspective and also orthologique, i.e., the perpendiculars from the angular points of one triangle on the corresponding sides of the other are concurrent, show that the centre of perspective is collinear with the two orthologique centres.

Solution and Remarks by K. Rangaswami.

Suppose that two triangles Δ_1, Δ_2 are so related to a conic Ω that Δ_1 and the polar-triangle Δ_2' of Δ_2 with respect to Ω are in perspective; there is a conic Ω' such that Δ_1, Δ_2' are polar triangles with respect to Ω', and we can pass from Δ_1 to Δ_2 by successive reciprocations with respect to Ω' and Ω. This amounts to a linear transformation Γ having for fixed points the vertices of the common self conjugate triangle of Ω, Ω'.

In particular, when Ω is the absolute conic, Γ is an affine transformation whose finite invariant lines are perpendicular. Further if Δ_1, Δ_2 are in perspective, the transformation Γ is *central* so that pairs of corresponding points are collinear with the centre of respective. Hence the two orthologique centres, being corresponding points in Γ, are collinear with the centre of perspective. A similar result holds good for two tetrahedra.

Also solved analytically by S. Narayanan.

Question 1705

(N. K. NARASIMHAMURTI):—Show that the star (α, δ) will have a heliacal rising when the right Ascension of the sin is R, where

$$\tan R = \frac{\tan\delta\cos\alpha - \sin\alpha\sqrt{\cot^2\phi - \tan^2\delta}}{\tan\omega - \tan\delta\sin\alpha - \cos\alpha\sqrt{\cot^2\phi - \tan^2\delta}}.$$

Hence deduce that

(1) at the equator, the heliacal rising of a star occurs when $R = \alpha$.

(2) at the poles, when the star is on the celestial equator, and the sun on the horizon, the star has a heliacal rising.

(3) an equatorial star will have a heliacal rising when R is given by $\tan R = \dfrac{\sin \alpha \cot \phi}{\cos \alpha \cot \phi - \tan \omega}$.

(4) The first point of Aries has a heliacal rising, if the co-latitude of the place is equal to the obliquity of the ecliptic. Then, the first point of Aries rises with the sun daily during half the year, and the first point of Libra during the other half.

Solution and Remarks by T. Totadri Iyengar.

Let P be the pole, S the sun, σ the star, and n the north point. Let Δ be the declination of the sun. For heliacal rising S, σ are in the eastern horizon. Denoting angles S Pn and σPn by ψ, θ respectively, from the right-angled spherical triangles SPn and σPn, we obtain

$$\cos \psi = \tan \phi \tan \Delta = \sin R \tan \omega \tan \phi \text{ where } \omega = \text{obliquity}$$

$$\cos \theta = \tan \phi \tan \delta.$$

Also, $\psi - \theta = \alpha - R$. i. e. $\psi = \theta + (\alpha - R)$

$$\therefore \quad \cos \psi = \cos \theta \cos (\alpha - R) - \sin \theta \sin (\alpha - R)$$

Substituting for $\cos \psi$ and $\cos \theta$, $\sin \theta$, we get

$$\sin R \tan \omega \tan \phi = \tan \phi \tan \delta \cos (\alpha - R) - \sqrt{1 - \tan^2 \phi \tan^2 \delta} \sin (\alpha - R).$$

$$\therefore \quad \sin R \tan \omega = \tan \delta \cos (\alpha - R) - \sqrt{\cot^2 \phi - \tan^2 \delta} \sin (\alpha - R).$$

On rearranging and dividing by $\cos R$, we get the value of $\tan R$ as desired.

Remarks

When the point (R, Δ) is rising, it is obvious that the point $(180^\circ + R, -\Delta)$ is setting. The answer above given is unaffected by changing R into $180^\circ + R$. Hence the answer furnishes also the right ascension of the sun, corresponding to star-rise at sun-set.

(1) At the equator $\phi = 0$. On substitution, the value of $\tan R$ becomes of the form ∞/∞ and on evaluating it as a limit for $\phi \longrightarrow 0$, we obtain $\tan R = \tan \alpha$. Heliacal rising occurs when $R = \alpha$. The solution $R = 180^\circ + \alpha$, corresponds to star-rise at sun-set.

(2) At the poles $\phi = \pm \pi/2$. The value of R will be unreal unless the radical which now becomes $\sqrt{-\tan^2 \delta}$ vanishes. i. e. $\delta = 0$. Thus the star must be on the equator (which in the present case is obvious as the horizon is the equator). The sun will be on the horizon only on March 21st or September 23rd.

On these dates, the star will rise heliacally only if its Right Ascension does not lie between 6 hrs. and 18 hrs. This is obvious from a figure. If its right ascension lies between 6 hrs. and 18 hrs. the star rises at sun-set. The question has thus to be slightly modified.

(3) Putting $\delta = 0$, we get at once

$$\tan R = (\sin \alpha \cot \phi)/(\cos \alpha \cot \phi - \tan \omega)$$

(4) The question is not carefully worded.

The first point of Aries γ has always heliacal rising on the two dates March 21st and September 23rd whatever ϕ be. This is also obtained by putting in (3) above, $\alpha = 0$. In the special case $\phi = \pi/2 - \omega$, both numerator and denominator become zero, the value of tan R becoming completely indeterminable. In this case the Equator and Horizon intersect at an angle ω, and when γ rises, the Horizon becomes the Ecliptic. Hence between winter solstice and summer solstice, there is heliacal rising of γ. During the other six months the sun sets when γ rises i.e. the sun rises with Libra.

Also solved by K. Subramaniam and M. V. Vaidyanatha Sastri.

Question 1712

(A. RANGANATHA RAO) :—The normals at four points ABCD on a conic S are concurrent at O. Prove that the directrix of the parabola which touches the tangents to S at ABCD passes through O and that the parabola is impolar to S.

Solution by S. Narayanan.

The feet of the normals from O to S lie on the Apollonian rectangular hyperbola which passes through O and the centre of S, and whose asymptotes are parallel to the axes of S. Reciprocate with respect to S. The tangents to S at A, B, C, D touch a parabola which also touches the axes of S and whose directrix is the line joining O to the centre of S.

The triangle formed by the axes of S and the line at infinity is self-conjugate to S and inscribed in the rectangular hyperbola. Hence the reciprocal parabola is inscribed in this triangle, and therefore it is impolar to S.

Also solved by T. K. Raghavachari and F. C. Auluck.

BOOKS RECEIVED FOR REVIEW

B. B. BAGI: *The Analytical Geometry of Conic Sections*, Dharwar, pp. 248, 1936.

Booklets of the series: Actualités Scientifiques et Industrielles, Herman et Cie, Paris:

No. 331. NICOLESCO: *Les fonctions polyharmoniques.*

No. 356. G. DARMOIS: *Methodes dé estimation.*

No. 357. SANTALO, L. A.: *Integralgeometrie 5.*

No. 358. CARTAN, E.: *La Topologie des groupes de Lie.*

No. 362. ORE, O.: *L' Algebre Abstraite.*

No. 427. PLATRIER: *Les axioms de la mechanique Newtonienne.*

No. 436. POSSEL, René de: *Jeux de Hasard et de Reflexion.*

WEYRICH: *Die Zylinderfunctionen und ihre Anwendungen;* B. G. Teubner, Price 5·70 R.M. with paper cover.

REVIEWS

CLAUDE CHEVALLY: *L' Arithmétique dans les algèbres de matrices*

[Actualités Scientifiques et Industrielles No. 323, Herman et Cie, Paris, (1936)]

This little volume of about thirty pages by a well-known investigator belonging to the new school of Modern Algebra is not an exposition of any known topic of the subject but is to be considered as an original memoir by the author. The fundamental problem of modern algebra is the formulation of the arithmetics of algebras and the extraordinary developments in the theory of ideals during the last decade have made this formulation possible in several important types of algebras. The theories of hyper complex arithmetic of Speiser, Brandt, Artin, and Hassé have shown that in a simple algebra of which the nucleus is a domain of algebraic numbers, a definition of an arithmetic is possible. The author obtains in this memoir a similar result for a complete algebra of matrices. He has shown that, if C be such an algebra of matrices whose coefficients belong to a domain K (commutative or not) and if one can define in K a regular arithmetic, then it is also possible to define a regular arithmetic in C.

The author first defines the notion of an algebra admitting of a "regular arithmetic" as being equivalent to the definition of a system of ideals possessing

specified properties, and shows that in such an arithmetic the classical properties of ideals are valid. The relation between the arithmetics of K and C is brought about by introducing the modulus of the linear forms of n variables with coefficients in K and considering this a right C-modulus and it is shown that C is inversely isomorphous with the ring of K-automorphisms of the modulus. An interesting theorem on the structure of the finite o-moduli of rank n (where o is the maximum order of K) is also deduced.

The complete investigation is carried out for the case where K is commutative but is capable of being easily extended to the non-commutative case also. In Appendix I the author deals briefly with the latter general case. The second appendix is entirely foreign to the subject-matter of the memoir and contains the exposition of a new method in the theory of elementary divisors.

B. S. M.

CH. PLATRIER: *Exposés de Géométrie Cinématique.*

[Actualités Scientifiques et Industrielles, Nos. 325–327].

I. *Cinématique du solide et théorie des vecteurs.*

II. *La masse en cinématique et théorie des tenseures du second ordre.*

III. *Cinématique des milieux continus.*

These three volumes of the Actualités series are devoted to an exposition of Kinematics preparatory to a study of Newtonian mechanics and are founded on a course of lectures delivered by the author at the 'Ecole Polytechnique' and the 'Ecole Nationale des Ponts et Chaussees'. The exposition in all three volumes is quite elementary and treats in a comprehensive and novel way several fundamental kinematic problems. The use of cartesian tensors in the second volume for treating questions connected with centre of gravity and moments of inertia serves as an excellent introduction to this important tool of investigation in modern theoretical Physics.

I. The first volume starts off with very elementary vector algebra and even at this stage is introduced the notion of central axis of a system of vectors. This is followed by a short second chapter on particle kinematics wherein the notions of a trajectory, velocity, acceleration, hodographs and deviation are briefly explained. The last three chapters are devoted to the kinematics of a rigid body starting from the elementary notions of translation and rotation and developing the classical theorems on the composition of infinitesimal motions. The nature of the most general type of finite movement of a rigid body is discussed in the last chapter and the author derives some very interesting theorems connecting such motions with the theory of roulettes and glissets.

II. The second volume introduces the notion of mass kinematically, and deals with the allied topics of centre of inertia, moments of inertia, momentum, kinetic energy and energy of acceleration the last named being the important quantity in terms of which Appell's equations of motions in general mechanics are expressed. The third chapter is devoted to a development of cartesian tensors of the second order with special reference to symmetric tensors. The tensor gradient of a vector and the vector divergence of a tensor are defined and the importance of these in mechanics is clearly exhibited. An immediate

application of tensors is made in the next chapter to the subject of moments of inertia, the ellipsoid of inertia and its principal axes. A chapter is devoted to moving axes and its applications to the kinematics of a rigid body, especially questions of angular momentum. The kinetic energy of a rigid body is treated in Chapter VIII and the notion of a principal solid as the rigid body of minimum kinetic energy is introduced to derive the principal theorem of Poincaré-Le Roux. The last chapter on the energy of acceleration contains much material not ordinarily treated in text-books and contains a detailed treatment of Koenig's theorem on the expression of this energy as the sum of the energy of acceleration of the centre of gravity and the energy of the system about the centre of gravity. Defining 'solides privilégiés' as those solids with reference to which the quantities of acceleration of a given system form a system of nul vectors the author derives interesting theorems about such solids. These theorems and the use of the tensor notation make it possible to establish a close connection between the theorems of Koenig and Poincaré-Le Roux.

III. The third and the smallest volume concerns itself with the kinematics of deformable media and serves as an elementary introduction to hydrodynamics. The hypothesis of continuity is explained and a series of obvious consequences are deduced thereby. The displacement vector, the symmetric translation tensor, the deformation tensor and the ellipsoid of dilatations are briefly defined and explained. The important theorem on the decomposition of an infinitesimal transformation into a translation, a rotation and a deformation is proved in the second chapter and a detailed study is made of the nature of a pure deformation. The last chapter treats of the subject proper of the continuous motion of a fluid medium by the methods of Euler and Lagrange and derives the equation of continuity.

B. S. M.

L. GODEAUX: *Les Involutions cycliques appartenant à une surface Algebrique.* [Actualités Scientifiques et Industrielles, No. 270, Hermann et Cie, Paris]. pp. 43, price 12 francs.

This booklet comprises the results of the author concerning the involutions of any algebraic surface possessing a birational transformation of finite order into itself, possessing only a finite number of coincident points. The last restriction is not so narrow as it may appear at first sight. If we take a specific example say, that of an ordinary plane, then all transformations (excepting homology—discussion in its case being easy) of the type considered, possess only 3 coincident points.

The author discusses the relation between the original surface F and ϕ, the image of the involution in relation to the coincident points etc. He constructs suitable models of F and ϕ in such a way that the involution is generated by means of a cyclic homology on the model of F. He classifies the coincident points into perfect and imperfect classes according as they satisfy particular properties and also classifies the involutions into regular and irregular involutions.

There is a valuable and exhaustive bibliography at the end concerning the transformation of algebraic surfaces.

K. V. I.

GOURSAT : *Propertiés generals de l'equation d'Euler et de Gauss.*

[Actualités Scientifiques et Industrielles, No. 333, Hermann et Cie, Paris 1936] pp. 92, price 20 francs.

This is an elementary monograph on the hypergeometric function. It possesses the lucidity and simplicity which characterises the other works of the author.

Only an elementary knowledge of the theory of linear differential equations is necessary for understanding the book. The five chapters of the book deal respectively with (i) the 24 solutions of Kummer etc , (ii) Solution by quadratures, (iii) The irreducible cases, (iv) The important concept of the automorphic group of a hypergeometric equation, (v) The linear relations between integrals.

The author has made a very judicious selection of the enormous number of results connected with the hypergeometric function and the book under review offers a very good introduction to the theory.

K. V. I.

P. DUBRIEL : *Quelques propertiés des varietés.*

[Actualités Scientifiques et Industrielles, No. 210, Hermann et Cie, Paris]. pp. 30, price 10 francs.

This is a small monograph published in memory of J. Herbrand who died at a very early age while still engaged in important researches in the field of modern algebra.

The author considers the relation between the decomposition of a polynom-ideal into primary factors [i.e. considering it as the least common multiple of a set of primary ideals—viz. Lasker's theorem] and some connected geometrical results. In the first chapter he gives some criteria for the non-existence of an improper component in the resolution. In the next chapter (viz. the last) he treats of some geometrical theorems as Severi's extension of the classical theorem of Nöether concerning the intersection of plane curves. He also points out the correlation between mixed intersections (e.g. surfaces having a curve and a number of isolated points common) of surfaces and the prevalence of an improper component in the resolution of the homogeneous ideal formed by the forms corresponding to the surfaces as basis into primary ideals.

K. V. I.

A. C. MUKHERJI : *Étude Statistique de la Fécondité Matrimoniale*

[Actualités Scientifiques et Industrielles, Hermann et Cie, Paris, 1935].

Studies on human fertility have now attracted great attention partly because fertility, not mortality is the decisive factor in determining population, and partly on account of the fact that the gap between fecundity and fertility is widening more and more. But the means of measuring ferfility are not as adequate or efficient as, say, those of measuring mortality. The simple ratio of

births to marriages is defective because of the complications arising out of second marriages, illegitimate births, the numerator of the ratio not being related to the marriages which produced the births, and so on. The conception of specific fertility rates giving the number of daughters born to a marriage for each year of life throughout the child-bearing period of women is applied and a *gross* reproductive rate is obtained It will be noted that if this is less than unity, the population cannot maintain itself however low may be the death rate. If, however, we determine the average number of girls born to each woman during the child-bearing period taking account both of fertility and mortality, we obtain a measure of *net* reproductivity. A study from this point of view further improved on account of divorces, has enabled the author of this dissertation, M. Mukherji, to introduce a new coefficient, the mean interval between two confinements. This he has applied to the statistics of France with, as his Editor, Professor Huber, says, "beaucoup d'ingéniosité." As a reward for this ingenuity the author was awarded the Doctorate degree of the Paris University. This simple and beautiful method is however not applicable even to the statistics of England and Wales because the age of the mother is not recorded on the birth certificate. But in such cases (and Indian Statistics are of course even more imperfect) it is also possible to estimate the distribution of births by age of mother from other sources of information and to calculate therefrom an estimated net reproduction rate. Such estimates for India are of great interest at the moment, and it is to be hoped that with the opportunities now available to the author as the Director of Statistics in a progressive Indian State, these gaps in international comparisons will be filled.

K. B. MADHAVA

QUESTIONS FOR SOLUTION

1667. (Corrected) (A. A. KRISHNASWAMI AYYANGAR).

Prove that $\int_0^x e^{-\frac{1}{2}x^2}\,dx > \sqrt{\pi/2}\,(1-e^{-x^2})$

when $1 > x > 0$.

(iii) $\int_0^x e^{-\frac{1}{2}x^2} dx < \sqrt{\pi/2}\,(1 - e^{-mx^2})$

when $x > \alpha_m$.

1740. (ALFRED MÖESSNER)—

Show how to obtain solutions in integers of the system of equations*

$$(A^2,\ B^2,\ A^2+B^2)^2 = (C,\ D,\ C+D)^2$$
$$(A^2,\ B^2,\ A^2+B^2)^4 = (C,\ D,\ C+D)^4$$

Example

A = 3,	B = 4,	C = 5,	D = 19
A = 5,	B = 12,	C = 59,	D = 120
A = 7,	B = 24,	C = 280,	D = 411
A = 21,	B = 28,	C = 21,	D = 1064

1741. (C. N. SRINIVASIENGAR)—Required a *purely geometrical* proof of the following :—The planes through the line of intersection of two of three given planes and perpendicular to the third intersect along a line.

Also show how the result can be generalised in the case when the three given planes intersect at a finite point.

1742. (K. RANGASWAMI) :—Through the mid-point E of the perpendicular CD, drawn from the right angle C of a right-angled triangle ABC, on AB lines are drawn parallel to the sides of the triangle. Show that :

(i) The six points of intersection of these lines with the sides of the triangle lie on a circle.

(ii) As C moves on the circum-circle of ABC, the centre of the circle through the six points describes an ellipse whose eccentricity is ½. Investigate the positions and lengths of the axes of the ellipse.

1743. (N. DORAIRAJAN)—If P be any point on the edge CD of a